

A LEVEL MATHEMATICS

SEQUENCES AND SERIES

NAME _____

Answers based on calculator technology are not acceptable

Question 1

A sequence $u_1, u_2, u_3, \dots$ is defined by

$$u_{n+1} = 3(u_n)^2 - 1$$

$$u_1 = k$$

where k is a real constant.

(a) Show that

$$u_3 = 27k^4 - 18k^2 + 2$$

(3)

Given that

$$u_3 - 5u_2 = 11$$

(b) find the possible values of k .

(a) $u_1 = k$
 $u_2 = 3k^2 - 1$
 $u_3 = 3(3k^2 - 1)^2 - 1$
 $= 3(9k^4 - 6k^2 + 1) - 1$
 $= \underline{27k^4 - 18k^2 + 2}$

(b) $27k^4 - 18k^2 + 2 - 5(3k^2 - 1) = 11$
 $27k^4 - 18k^2 + 2 - 15k^2 + 5 = 11$
 $27k^4 - 33k^2 - 4 = 0$
 $(3k^2 - 4)(9k^2 + 1) = 0$

$\Rightarrow k^2 = \frac{4}{3}$, $k^2 = -\frac{1}{9}$
 $k = \pm \frac{2}{\sqrt{3}}$ Invalid

Question 2

An arithmetic series has first term a and common difference d , where a and d are constants.

Given that the sum of the first 65 terms of the series is 4940

(a) show that

$$a + 32d = 76 \quad (2)$$

Given also that the 49th term of the series is 64

(b) write down another equation connecting a and d . (1)

(c) Hence find the value of a and the value of d . (2)

Given that S_n represents the sum of the first n terms of the series,

(d) find the largest possible positive value of S_n

(Solutions relying entirely on calculator technology are not acceptable.)

(a) $S_n = \frac{n}{2} [2a + (n-1)d]$
 $4940 = \frac{65}{2} [2a + 64d]$
 $152 = 2a + 64d$
 $76 = a + 32d$ shown.

(b) $U_n = a + (n-1)d$
 $64 = a + 48d$

(c) $76 = a + 32d$
 $64 = a + 48d$

 $12 = -16d$
 $-\frac{3}{4} = d$
 $a = 100$

(d) $U_n = 100 + (n-1)\left(-\frac{3}{4}\right)$
 $0 = 100 - \frac{3}{4}n + \frac{3}{4}$
 $\frac{3}{4}n = 100.75$
 $n = 134.\dot{3}$

$S_{134} = \frac{134}{2} [2(100) + 133\left(-\frac{3}{4}\right)]$
 $= \underline{6716.75}$

Question 3

In this question you must show all stages of your working.
Solutions relying on calculator technology are not acceptable.

The 3rd, 4th and 5th terms of a geometric sequence are $\sin \theta$, $\sqrt{2} \cos \theta$ and $\sqrt{3} \cot \theta$ respectively, where θ is a constant, $0 < \theta < \frac{\pi}{2}$

(a) Find the exact value of the common ratio of this sequence.

(4)

(b) Hence find, in simplest form, the exact value of the sum of the first 3 terms of the sequence.

(3)

$$(a) \frac{\sqrt{3} \cot \theta}{\sqrt{2} \cos \theta} = \frac{\sqrt{2} \cos \theta}{\sin \theta}$$

$$\sqrt{3} \sin \theta \times \frac{\cos \theta}{\sin \theta} = 2 \cos^2 \theta$$

$$\sqrt{3} \cos \theta = 2 \cos^2 \theta$$

$$2 \cos^2 \theta - \sqrt{3} \cos \theta = 0$$

$$\cos \theta (2 \cos \theta - \sqrt{3}) = 0$$

$$\cos \theta = 0, \quad \cos \theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{2}, \quad \theta = \frac{1}{6}\pi$$

out of range.

$$r = \frac{\sqrt{2} \cos(\frac{\pi}{6})}{\sin \frac{\pi}{6}}$$

$$= \sqrt{6}$$

(b)

$$\text{---}, \text{---}, \frac{1}{2}, \frac{\sqrt{6}}{2}, 3$$

$$2^{\text{nd}} \text{ term} = \frac{1}{2} \div \sqrt{6} = \frac{\sqrt{6}}{12}$$

$$1^{\text{st}} \text{ term} = \frac{\sqrt{6}}{12} \div \sqrt{6} = \frac{1}{12}$$

$$\text{Sum} = \frac{1}{12} + \frac{\sqrt{6}}{12} + \frac{1}{2}$$

$$= \frac{7}{12} + \frac{\sqrt{6}}{12}$$

Question 4

A sequence of terms $a_1, a_2, a_3, \dots$ is defined by

$$\begin{aligned}a_1 &= 4 \\ a_{n+1} &= ka_n + 3\end{aligned}$$

where k is a constant.

Given that

- $\sum_{n=1}^3 a_n = 12$
- all terms of the sequence are different

find the value of k

(4)

$$a_1 = 4$$

$$a_2 = 4k + 3$$

$$\begin{aligned}a_3 &= k(4k + 3) + 3 \\ &= 4k^2 + 3k + 3\end{aligned}$$

$$\sum_{n=1}^3 a_n = 4 + 4k + 3 + 4k^2 + 3k + 3 = 12$$

$$\Rightarrow 4k^2 + 7k - 2 = 0$$

$$(4k - 1)(k + 2) = 0$$

$$\therefore k = \frac{1}{4} \quad \text{or} \quad k = -2$$

$$\text{If } k = \frac{1}{4}, \quad a_1 = a_2 = a_3 = 4$$

$$\therefore \underline{k = -2}$$

Question 5

In this question you must show all stages of your working.

Given that the first three terms of a geometric sequence are

$$8 \sin \theta$$

$$3 \sin 2\theta$$

$$2 + 2 \cos 2\theta$$

$$\text{and } 0 < \theta < \frac{\pi}{2}$$

(a) show that

$$\sin \theta = \frac{8}{9}$$

Hence, or otherwise

(b) prove that S_{∞} exists

(c) find the value of $S_{\infty} - S_{10}$ giving your answer to 2 significant figures.

$$\begin{aligned} (b) \quad r &= \frac{6 \sin \theta \cos \theta}{8 \sin \theta} \\ &= \frac{3 \cos \theta}{4} \\ &= \frac{\sqrt{17}}{12}, \quad |r| < 1 \\ &\therefore S_{\infty} \text{ exists} \end{aligned}$$

(4)

$$(c) \quad a = 8 \sin \theta = \frac{64}{9}$$

$$S_{\infty} = \frac{a}{1-r}, \quad S_{10} = \frac{a(1-r^{10})}{1-r}$$

(2)
(3)

$$\begin{aligned} S_{\infty} - S_{10} &= \frac{ar^{10}}{1-r} \\ &= \frac{\frac{64}{9} \times \left(\frac{\sqrt{17}}{12}\right)^{10}}{1 - \frac{\sqrt{17}}{12}} \\ &= 2.5 \times 10^{-4} \end{aligned}$$

$$\begin{aligned} (a) \quad \frac{2 + 2 \cos 2\theta}{3 \sin 2\theta} &= \frac{3 \sin 2\theta}{8 \sin \theta} \\ \frac{2 + 2(2 \cos^2 \theta - 1)}{3 \times 2 \sin \theta \cos \theta} &= \frac{3 \times 2 \sin \theta \cos \theta}{4 \times 8 \sin \theta} \\ \frac{2 + 4 \cos^2 \theta - 2}{6 \sin \theta \cos \theta} &= \frac{3 \cos \theta}{4} \\ \frac{4 \cos^2 \theta}{6 \sin \theta \cos \theta} &= \frac{3 \cos \theta}{4} \\ \frac{2 \cos \theta}{3 \sin \theta} &= \frac{3 \cos \theta}{4} \\ \frac{3 \sin \theta}{2 \cos \theta} &= \frac{4}{3 \cos \theta} \\ 3 \sin \theta &= \frac{4 \times 2 \cos \theta}{3 \cos \theta} \end{aligned}$$

$$3 \sin \theta = \frac{8}{3}$$

$$\boxed{\sin \theta = \frac{8}{9}}$$

Question 6

The first three terms of an arithmetic sequence are

$$6k, 10 \text{ and } 2k$$

where k is a constant.

(a) Find the value of k .

(2)

(b) Hence find the value of the sum of the first 50 terms of this sequence.

(3)

$$(a) \quad 10 - 6k = 2k - 10$$

$$20 = 8k$$

$$2.5 = k$$

$$(b) \quad S_n = \frac{n}{2} [2a + (n-1)d] \quad \begin{cases} a = 6 \times 2.5 = 15 \\ d = -5 \end{cases}$$

$$\begin{aligned} S_{50} &= 25 [30 + 49 \times (-5)] \\ &= \underline{\underline{-5375}} \end{aligned}$$

Question 7

A sequence $u_1, u_2, u_3 \dots$ is defined by

$$u_1 = 35$$

$$u_{n+1} = u_n + 7 \cos\left(\frac{n\pi}{2}\right) - 5(-1)^n$$

(a) (i) Show that $u_2 = 40$

(ii) Find the value of u_3 and the value of u_4

(3)

Given that the sequence is periodic with order 4

(b) (i) write down the value of u_5

(ii) find the value of $\sum_{r=1}^{25} u_r$

(3)

(a) (i) $u_1 = 35$

$$\begin{aligned} u_2 &= 35 + 7 \cos\left(\frac{\pi}{2}\right) - 5(-1) \\ &= 35 + 5 = 40 \text{ shown.} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad u_3 &= 40 + 7 \cos(\pi) - 5 \\ &= 28 \end{aligned}$$

$$\begin{aligned} u_4 &= 28 + 7 \cos\left(\frac{3\pi}{2}\right) - 5(-1) \\ &= 33 \end{aligned}$$

b(i): 35, 40, 28, 33, because
period is order 4,
 $u_1 = u_5 = 35$

(ii) $\sum_{r=1}^{25} u_r$: One complete period has
Sum $35 + 40 + 28 + 33$
 $= 136$

So $\sum_{r=1}^{25} u_r$ There are $6(4) + 1$
 $= 6(136) + 35$
 $= 816 + 35 = \boxed{851}$

Question 8

The first three terms of a geometric sequence are

$$3k + 4 \quad 12 - 3k \quad k + 16$$

where k is a constant.

(a) Show that k satisfies the equation

$$3k^2 - 62k + 40 = 0$$

(2)

Given that the sequence converges,

(b) (i) find the value of k , giving a reason for your answer,

(ii) find the value of S_{∞} .

(5)

$$(a) \frac{k+16}{12-3k} = \frac{12-3k}{3k+4}$$

$$(k+16)(3k+4) = (12-3k)(12-3k)$$

$$3k^2 + 52k + 64 = 144 - 72k + 9k^2$$

$$6k^2 - 124k + 80 = 0$$

$$\frac{6k^2 - 124k + 80}{2} = 0$$

$$3k^2 - 62k + 40 = 0$$

$$b(i) 3k^2 - 62k + 40 = 0$$

$$k = \frac{62 \pm \sqrt{(-62)^2 - 4(3)(40)}}{6}$$

$$k = 20 \quad \text{or} \quad \frac{2}{3}$$

$$\text{When } k=20: 64, -48, 36 \therefore r = -\frac{3}{4}$$

$$|r| = \frac{3}{4} < 1 \therefore \text{converges}$$

$$\text{When } k = \frac{2}{3}: 6, 10 \therefore r = \frac{5}{3}$$

$$|r| = \frac{5}{3} > 1 \therefore \text{does not converge.}$$

$$(c) S_{\infty} = \frac{a}{1-r}$$

$$= \frac{64}{1 - (-\frac{3}{4})}$$

$$= \frac{256}{7}$$